EPFL– Fall 2025
Series 4

Differential Geometry IV: General Relativity

G. Moschidis
1 Oct. 2025

4.1 Let $\mathcal{M}^n$ be a differentiable manifold.

(a) Show that, for any $X, Y, Z \in \Gamma(\mathcal{M})$:

$$\mathcal{L}_{[X,Y]}Z = \mathcal{L}_X\mathcal{L}_Y Z - \mathcal{L}_Y\mathcal{L}_X Z.$$

Show that the above relation also holds when Z is replaced by any tensor field f of type (k,l), $k,l \in \mathbb{N}$. *(Hint: Check how $\mathcal{L}_X$ behaves on tensor products of the form $f_1 \otimes f_2$.)*

(b) Show that the space $\mathcal{K}$ of Killing vector fields on $(\mathcal{M}, g)$ is closed under commutation, i.e. that $[X,Y] \in \mathcal{K}$ if $X, Y \in \mathcal{K}$; thus, $\mathcal{K}$ forms a Lie subalgebra of $\Gamma(\mathcal{M})$.

(c) Let g be a Lorentzian metric on $\mathcal{M}$. Let X be a Killing field on $(\mathcal{M}, g)$. Show also that, for any $V, W \in \Gamma(\mathcal{M})$:

$$g\big(\nabla_V X, W\big) + g\big(\nabla_W X, V\big) = 0,$$

where ∇ denotes the Levi-Civita connection associated to g (recall that ∇ is torsion-free and satisfies $\nabla_X g = 0$ for all $X \in \Gamma(\mathcal{M})$.) (*Hint: Apply the product rule on the expression $X(g(Y,Z)) = \mathcal{L}_X(g(Y,Z))$ for suitably chosen vector fields Y, Z.*)

4.2 (a) Let $F : (\mathbb{R}^{n+1}, \eta) \to (\mathbb{R}^{n+1}, \eta)$ be an isometry of Minkowski spacetime. Show that, with respect to the Cartesian coordinate system $(x^0, \ldots, x^n)$ on $\mathbb{R}^{n+1}$, the differential dF satisfies at every $x \in \mathbb{R}^{n+1}$:

$$\eta_{\mu\nu}\partial_\alpha F^\mu(x)\partial_\beta F^\nu(x) = \eta_{\alpha\beta}.$$

Deduce that F is an *affine* isometry, i.e. takes the form $F(x) = Ax + b$. (*Hint: Differentiate the above relation.*)

(b) Let $t \to F_t : (\mathbb{R}^{n+1}, \eta) \to (\mathbb{R}^{n+1}, \eta)$ be a 1-parameter group of isometries. Prove that the generator of $\{F_t\}_{t\in\mathbb{R}}$, namely the vector field

$$x \to X|_x \doteq \frac{d}{dt}F_t(x)\Big|_{t=0},$$

is *affine*, i.e. is of the form

$$X^\mu(x) = A^\mu_\nu x^\nu + b^\mu$$

for some constants A^μ_ν, b^μ. Show also that the matrix A^μ_ν satisfies

$$\eta_{\alpha\mu}A^\mu_\beta + \eta_{\beta\mu}A^\mu_\alpha = 0.$$

What is the dimension of the Lie algebra of Killing vector fields on $(\mathbb{R}^{n+1}, \eta)$?

(c) Can you similarly classify all conformal Killing vector fields on $(\mathbb{R}^{n+1}, \eta)$ when $n \geqslant 2$?

(d) Show that the Lie group of isometries of the n-dimensional de Sitter space (see Exercise 2.1) has dimension at least $\frac{n(n+1)}{2}$ (we will later show that this is also an upper bound for the dimension of the group of isometries of any n-dimensional Lorentzian manifold).

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4.3 In this exercise, we will study the question of whether a given Lorentzian manifold can be extended, i.e. isometrically embedded into a larger Lorentzian manifold in a non trivial way. This question arises often in general relativity in the study of singularities emerging from smooth "initial data" (such as the singularities in the interior of black holes, or "big bang"-type singularities).

(a) Consider the spacetime $(\mathcal{M}, g)$ with $\mathcal{M} = \mathbb{R}_t \times (x_0, +\infty)_x$, $x_0 > 0$, and

$$g = -\big(1 - \frac{x_0}{x}\big)dt^2 + \big(1 - \frac{x_0}{x}\big)^{-1}dx^2.$$

Show that $(\mathcal{M}, g)$ can be extended smoothly "beyond $x = x_0$" by finding a coordinate transformation $(t, x) \to (\tilde{t}(t, x), x) = (t + f(x), x)$ such that, with respect to the new coordinates, the components of g can be smoothly extended as functions of $(\tilde{t}, x)$ beyond $x = x_0$. Can you express this process as an embedding of $(\mathcal{M}, g)$ into a larger Lorentzian manifold?

*(b) Consider the spacetime $(\mathcal{N}, h)$ with $\mathcal{N} = (0, +\infty)_t \times \mathbb{S}^1_\theta$ and

$$h = -t dt^2 + \frac{1}{t} d\theta^2.$$

We will show that $(\mathcal{N}, h)$ is inextendible as a C^0 Lorentzian manifold beyond $t = 0$. We will achieve this in a number of steps; we will essentially follow the method introduced by Sbierski to prove a similar statement for the interior of the Schwarzschild black hole.

1. For any Lorentzian manifold $(\mathcal{M}', g')$ and any domain $\mathcal{U}$, we will define the *spacelike diameter* of $\mathcal{U}$ by

$$\mathrm{spdiam}\mathcal{U} \doteq \sup\Big\{\ell(\gamma) : \gamma \subset \mathcal{U} \text{ is a spacelike curve }\Big\}.$$

Show that if $p, q \in (\mathbb{R}^{1+1}, \eta)$ with $q \in I^-(p)$, then

$$\mathrm{spdiam} I^+(q) \cap I^-(p) \leqslant \sqrt{(p^0 - q^0)^2 - (p^1 - q^1)^2}.$$

2. If g is a C^0 oriented Lorentzian metric on $\mathbb{R}^{1+1}$, show that, for every pairs of sequences of points $p_n, q_n \in \mathbb{R}^n$ with $q_n \in I^-(p_n)$ (the past cone defined with respect to g) and $\lim_{n\to+\infty} q_n = \lim_{n\to+\infty} p_n = p$, we have

$$\mathrm{spdiam} I^+(q_n) \cap I^-(p_n) \xrightarrow{n\to+\infty} 0.$$

3. We will assume as given the following statement:
If $(\mathcal{M}', g')$ is a $1+1$ dimensional and oriented C^0 Lorentzian manifold and $\Omega \subset \mathcal{M}'$ is a domain in $\mathcal{M}'$ with the property that Ω covers the past of $\partial\Omega$, i.e. $I^-(p) \subset \Omega$ for any $p \in \partial\Omega$, then the following holds: For any $p \in \partial\Omega$, there exists a sequence $q_n \in I^-(p)$ and $p_n \in I^+(p)$ such that $q_n, p_n \xrightarrow{n\to+\infty} p$ and $I^+(q_n) \cap \Omega \subset I^-(p_n)$.
Using that, show that $(\mathcal{N}, h)$ above is not extendible as a C^0 spacetime beyond $t = 0$.